Level spacing statistics of classically integrable systems: Investigation along the lines of the Berry-Robnik approach

H. Makino[1,*] and S. Tasaki[2]

[1]*Department of Human and Information Science, Tokai University, C-216, 1117 Kitakaname, Hiratsuka-shi, Kanagawa 259-1292, Japan*

[2]*Department of Applied Physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan*

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By extending the approach of Berry and Robnik, the limiting level spacing distribution of a system consisting of infinitely many independent components is investigated. The limiting level spacing distribution is characterized by a single monotonically increasing function $\bar{\mu}(S)$ of the level spacing S. Three cases are distinguished: (1) Poissonian if $\bar{\mu}(+\infty)=0$, (2) Poissonian for large S, but possibly not for small S if $0<\bar{\mu}(+\infty)<1$, and (3) sub-Poissonian if $\bar{\mu}(+\infty)=1$. This implies that, even when energy-level distributions of individual components are statistically independent, non-Poissonian level spacing distributions are possible.

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I. INTRODUCTION

For a bounded quantum system in the semiclassical limit, statistical properties of energy levels at a given energy have been intensively studied. Universal behaviors are found in statistics of *unfolded* energy levels [1–3], which are sequence of numbers uniquely determined by the energy levels using the mean level density obtained from the Thomas-Fermi rule (see Appendix). For quantum systems whose classical counterparts are strongly chaotic, the unfolded level statistics are well characterized by the random matrix theory that gives level spacing distribution obeying the Wigner distribution [4,5].

For quantum systems whose classical counterparts are integrable (such systems will be referred to as classically integrable systems), level statistics were theoretically studied by Berry-Tabor [2], Sinai [6], Molchanov [7], Bleher [8], Connors and Keating [9], and have been the subject of many numerical investigations. Still its mechanism is not well understood, the level spacing distributions are believed to be the Poisson (exponential) distribution for generic classically integrable systems.

As suggested, e.g., by Hannay (see the discussion of Ref. [10]), one possible explanation would be as follows: For an integrable system of f degrees of freedom, almost every orbit is generically confined in each inherent torus, and whole region in the phase space is densely covered by invariant tori as suggested by the Liouville-Arnold theorem [11]. In other words, the phase space of the integrable system consists of infinitely many tori, which have infinitesimal volumes in Liouville measure. Then, the energy level sequence of the whole system is a superposition of subsequences, which are contributed from those regions. Therefore, if the mean level spacing of each independent subset is large, one would expect the Poisson distribution as a result of the law of small numbers [12].

As shown by the pioneering work of Berry and Robnik [10], this would be the case for the nearly integrable systems consisting of large number of independent components. Quantum energy eigenstates are considered to be ''a superposition'' of various classical trajectories, which are connected by the tunneling effect. In the semiclassical limit, because of the suppression of the tunneling, energy eigenstates are expected to be localized on individual region in phase space explored by a typical trajectory of classical system [13], such as tori and chaotic regions, and to form independent components. The level spacing distribution is, then, regarded as a product of superposition of energy levels contributed, respectively, from those independent components. Based on this view, Berry and Robnik observed that the level spacing distribution of the system consisting of a single regular component, described by a Poisson distribution, and N chaotic components, described by the Wigner distributions of equal strength, approaches the Poisson distribution when $N \to +\infty$.

The localization of energy eigenfunctions onto individual phase space structures was studied by Berry as a semiclassical behavior of the Wigner function [13]. For the classically integrable system, it is shown that the Wigner function tends to δ function on tori in the semiclassical limit [14]. Such localization phenomenon of eigenfunctions has been observed numerically for several systems [15–17]. For example, in Ref. [17], it was shown that the Husimi distribution function of each energy eigenstate is well localized either on the classical chaotic regions or on the regular regions filled with classical tori and that the energy eigenvalues corresponding to eigenfunctions supported by distinct phase-space regions obey distinct statistics.

Those works seem to imply that the existence of infinitely many independent components is an essential ingredient of the appearance of Poissonian level spacing distribution and that the semiclassical limit is one of the mechanisms providing infinitely many components. However, in some classically integrable or nearly integrable systems [2,8,18,19], where one might expect infinitely many components, deviations from the Poissonian distribution have been observed. Therefore, it is interesting to explore the consequences only

*Email address: makino@tokai.ac.jp

of the existence of infinitely many independent components.

In this paper, along the line of thoughts of Berry and Robnik [10], we investigate the distribution of unfolded energy levels when energy levels consist of infinitely many independent components, and show the possibility of deviation from the Poisson distribution. Hereafter, for the sake of simplicity, unfolded energy levels will be referred to as energy levels or levels.

We briefly review the Berry-Robnik theory [10]. It relates the statistics of the energy level distribution to the phase-space geometry by assuming that the sequence of the energy spectrum is given by the superposition of statistically independent subspectra, which are contributed, respectively, from eigenfunctions localized onto the invariant regions in phase space. Formation of such independent subspectra is a consequence of the condensation of energy eigenfunctions on disjoint regions in the classical phase space and of the lack of mutual overlap between their eigenfunctions, and, thus, can only be expected in the semiclassical limit where the Planck constant tends to zero, $\hbar \to 0$. This mechanism is sometimes referred to as *the principle of uniform semiclassical condensation of eigenstates* [20,21], which is based on an implicit state by Berry [13]. The validity of this assumption is checked numerically in the semiclassical (high energy) region where the Planck volume is much smaller than the phase volume of each invariant region [15,16,22].

In the Berry-Robnik approach [10], the overall level spacing distribution is derived along a line of mathematical framework by Mehta [4], as follows: Consider a system whose classical phase space is decomposed into N disjoint regions. The Liouville measures of these regions are denoted by ρ_i $(i=1,2,3,\ldots,N)$, which satisfy $\Sigma_{i=1}^{N}\rho_i=1$. Let $E(S)$ be the gap distribution function, which stands for the probability that an interval $(0,S)$ contains no level. $E(S)$ is expressed by the level spacing distribution $P(S)$ as follows:

$$E(S)=\int_S^\infty d\sigma \int_\sigma^\infty P(x)dx. \tag{1.1}$$

When the entire sequence of energy levels is a product of statistically independent superposition of N subsequences, $E(S;N)$ is decomposed into those of subsequences, $E_i(S;\rho_i)$,

$$E(S;N)=\prod_{i=1}^{N} E_i(S;\rho_i). \tag{1.2}$$

In terms of the normalized level spacing distribution $p_i(S;\rho_i)$ of subsequence, $E_i(S;\rho_i)$ is given by

$$E_i(S;\rho_i)=\rho_i \int_S^\infty d\sigma \int_\sigma^\infty p_i(x;\rho_i)dx, \tag{1.3}$$

and $p_i(S;\rho_i)$ is assumed to satisfy [10]

$$\int_0^\infty S p_i(S;\rho_i)dS=\frac{1}{\rho_i}. \tag{1.4}$$

Note that the spectral components are not always unfolded automatically in general even when the total spectrum is unfolded. However, in the sufficient small interval $[\epsilon,\epsilon+\Delta\epsilon]$, each spectral component obeys a same scaling law (see Appendix) and thus is unfolded automatically by an overall unfolding procedure. Equations (1.2) and (1.4) relate the level statistics in the semiclassical limit with the phase-space geometry.

In most general cases, the level spacing distribution might be singular. In such a case, it is convenient to use its cumulative distribution function μ_i:

$$\mu_i(S)=\int_0^S p_i(x;\rho_i)dx. \tag{1.5}$$

The corresponding quantity of the overall level spacing distribution is

$$M(S;N)=\int_0^S P(x;N)dx, \tag{1.6}$$

where $P(S;N)$ is the level spacing distribution function corresponding to $E(S;N)$.

In addition to Eqs. (1.2) and (1.4), we assume two conditions for the statistical weights.

Assumption (i). The statistical weights of independent regions uniformly vanishes in the limit of infinitely many regions,

$$\max_i \rho_i \to 0 \quad \text{as} \quad N \to +\infty. \tag{1.7}$$

Assumption (ii). The weighted mean of the cumulative distribution of energy spacing,

$$\mu(\rho;N)=\sum_{i=1}^{N} \rho_i \mu_i(\rho), \tag{1.8}$$

converges in $N \to +\infty$ to $\overline{\mu}(\rho)$

$$\lim_{N\to+\infty} \mu(\rho;N)=\overline{\mu}(\rho). \tag{1.9}$$

The limit is uniform on each closed interval: $0 \leqslant \rho \leqslant S$.

In the Berry-Robnik theory, the statistical weights of individual components are related to the phase volumes of the corresponding invariant regions. This relation is satisfactory if the Thomas-Fermi rule for the phase space fractions still holds, in general. Here we do not specify their physical meaning and deal with them as parameters.

Under Assumptions (i) and (ii), Eqs. (1.2) and (1.4) lead to the overall level spacing distribution whose cumulative distribution function is given by the following formula in the limit of $N \to +\infty$:

$$M_{\overline{\mu}}(S) = 1 - [1 - \overline{\mu}(S)] \exp\left[- \int_0^S [1 - \overline{\mu}(\sigma)] d\sigma \right], \tag{1.10}$$

where the convergence is in the sense of the weak limit. When the level spacing distributions of individual components are sparse enough, one may expect $\overline{\mu} = 0$ and the level spacing distribution of the whole energy sequence reduces to the Poisson distribution,

$$M_{\overline{\mu}=0}(S) = 1 - \exp(-S). \tag{1.11}$$

In general, one may expect $\overline{\mu} \neq 0$, which corresponds to a certain accumulation of the levels of individual components.

In the following sections, the above statement is proved and the limiting level spacing distributions are classified into three classes. One of them is the Poisson distribution as discussed in the original work by Berry and Robnik [10]. The others are not Poissonian. We give examples of non-Poissonian limiting level spacing distributions in Sec. III. In the concluding section, we discuss some relations between our results and other related works.

II. LIMITING LEVEL SPACING DISTRIBUTION

A. Derivation of the limiting level spacing distribution

In this section, starting from Eqs. (1.2) and (1.4), and Assumptions (i) and (ii) introduced in the preceding section, we show that, in the limit of infinitely many components $N \to +\infty$, the level spacing distribution converges weakly to the distribution with the cumulative distribution function:

$$M_{\overline{\mu}}(S) = 1 - [1 - \overline{\mu}(S)] \exp\left[- \int_0^S [1 - \overline{\mu}(\sigma)] d\sigma \right]. \tag{2.1}$$

According to Helly's theorem [12,23], this is equivalent to show that the cumulative distribution function $M(S;N)$ converges to $M_{\overline{\mu}}(S)$. The convergence is shown as follows.

Following the procedure by Mehta (see Appendix A.2 of Ref. [4]), we rewrite the gap distribution function $E(S;N)$ in terms of the cumulative level spacing distribution functions $\mu_i(S)$ of independent components:

$$\begin{aligned} E(S;N) &= \prod_{i=1}^N \left[\rho_i \int_S^{+\infty} d\sigma [1 - \mu_i(\sigma)] \right] \\ &= \prod_{i=1}^N \left[1 - \rho_i \int_0^S d\sigma [1 - \mu_i(\sigma)] \right]. \end{aligned} \tag{2.2}$$

The second equality follows from Eq. (1.4), integration by parts and $\lim_{\sigma \to +\infty} \sigma[1 - \mu_i(\sigma)] = 0$, which results from the existence of the average. Then, the overall cumulative level spacing distribution function $M(S;N)$ is given by

$$\begin{aligned} M(S;N) &= 1 + \frac{d}{dS} E(S;N) \\ &= 1 - E(S;N) \sum_{i=1}^N \frac{\rho_i - \rho_i \mu_i(S)}{1 - \rho_i \int_0^S d\sigma [1 - \mu_i(\sigma)]}. \end{aligned} \tag{2.3}$$

First we consider the behavior of $E(S;N)$. Since the convergence of $\Sigma_{i=1}^N \rho_i \mu_i(\sigma) \to \overline{\mu}(\sigma)$ for $N \to +\infty$ is uniform on each interval $\sigma \in [0,S]$ by Assumption (ii),

$$\begin{aligned} \log E(S;N) &= \sum_{i=1}^N \log\left[1 - \rho_i \int_0^S d\sigma [1 - \mu_i(\sigma)] \right] \\ &= - \sum_{i=1}^N \left[\rho_i \int_0^S d\sigma [1 - \mu_i(\sigma)] + O(\rho_i^2) \right] \\ &= - \int_0^S d\sigma [1 - \mu(\sigma;N)] + \sum_i^N O(\rho_i^2) \end{aligned} \tag{2.4}$$

$$\to - \int_0^S d\sigma [1 - \overline{\mu}(\sigma)] \quad \text{as} \quad N \to +\infty, \tag{2.5}$$

where we have used $|\mu_i(\sigma)| \leqslant 1$, $\log(1+\epsilon) = \epsilon + O(\epsilon^2)$ in $\epsilon \ll 1$, and the following property obtained from Assumption (i):

$$\left| \sum_{i=1}^N O(\rho_i^2) \right| \leqslant C \max_i \rho_i \sum_{i=1}^N \rho_i = C \max_i \rho_i \to 0 \quad \text{as} \quad N \to +\infty, \tag{2.6}$$

with C a positive constant. The $N \to +\infty$ limit of the sum in the right-hand side of Eq. (2.3) can be calculated in a similar way. Indeed, as $1/(1-\epsilon) = 1 + O(\epsilon)$ in $\epsilon \ll 1$, one has

$$\sum_{i=1}^N \frac{\rho_i - \rho_i \mu_i(S)}{1 - \rho_i \int_0^S d\sigma [1 - \mu_i(\sigma)]} = 1 - \sum_{i=1}^N \rho_i \mu_i(S) + \sum_{i=1}^N O(\rho_i^2) \tag{2.7}$$

$$\to 1 - \overline{\mu}(S) \quad \text{as} \quad N \to +\infty. \tag{2.8}$$

Therefore, we have the desired result:

$$\lim_{N\to\infty} M(S;N) = M_{\overline{\mu}}(S) = 1-[1-\overline{\mu}(S)] \times \exp\left[-\int_0^S [1-\overline{\mu}(\sigma)]d\sigma\right]. \quad (2.9)$$

B. Properties of the limiting level spacing distribution

Since $\mu_i(S)$ is monotonically increasing and $0 \leqslant \mu_i(S) \leqslant 1$, $\overline{\mu}(S)$ has the same properties. Then, $1-\overline{\mu}(S) \geqslant 0$ for any $S \geqslant 0$ and one has

$$\frac{1}{S}\int_0^S d\sigma[1-\overline{\mu}(\sigma)] \to 1-\overline{\mu}(+\infty) \quad \text{as} \quad S\to+\infty. \quad (2.10)$$

The limit classifies the following three cases:

Case 1, $\overline{\mu}(+\infty)=0$. The limiting level spacing distribution is the Poisson distribution. Note that this condition is equivalent to $\overline{\mu}(S)=0$ for $^{\forall}S$ because $\overline{\mu}(S)$ is monotonically increasing.

Case 2, $0<\overline{\mu}(+\infty)<1$. For large value of S, the limiting level spacing distribution is well approximated by the Poisson distribution, while, for small value of S, it may deviates from the Poisson distribution.

Case 3, $\overline{\mu}(+\infty)=1$. The limiting level spacing distribution deviates from the Poisson distribution for $\forall S$ in such a way that the cumulative distribution function approaches 1 as $S \to +\infty$ more slowly than does the Poisson distribution. This case will be referred to as a sub-Poisson distribution.

One has Case 1 if the individual level spacing distributions are derived from scaled distribution functions f_i as

$$\mu_i(S) = \rho_i \int_0^S f_i(\rho_i x)dx, \quad (2.11)$$

where f_i satisfy

$$\int_0^{+\infty} f_i(x)dx = 1, \quad \int_0^{+\infty} x f_i(x)dx = 1$$

and are uniformly bounded by a positive constant D: $|f_i(S)| \leqslant D$ $(1 \leqslant i \leqslant N$ and $S \geqslant 0)$. Indeed, one then has

$$|\mu(S;N)| \leqslant \sum_{i=1}^N \rho_i^2 \int_0^S |f_i(\rho_i x)|dx \leqslant DS\sum_{i=1}^N \rho_i^2 \leqslant DS \max_i \rho_i \sum_{i=1}^N \rho_i \to 0 \equiv \overline{\mu}(S). \quad (2.12)$$

This includes the case studied by Berry and Robnik [10], where the level spacing distribution is a superposition of a single regular component and N equivalent chaotic components, and the latter is expressed by the product of the scaled distributions as in Eq. (1.2). Indeed, one has

$$E^{BR}(S;N) = \exp(-\rho_0 S)\prod_{i=1}^N E_i^{\text{WIGNER}}(S;\rho_i), \quad (2.13)$$

where the statistical weights are $\rho_i=(1-\rho_0)/N$ and the individual level spacing distributions f_i corresponding to the gap distributions $E_i^{\text{WIGNER}}(S;\rho_i)$ are given by

$$f_i(x) = \frac{\pi x}{2}\exp\left[-\frac{\pi}{4}x^2\right]. \quad (2.14)$$

In addition, this would be the case when the system consists of identical N components where the level spacing distribution is described by a scaled form as Eq. (2.11). Such a case is expected when there is a symmetry, such as the regular polygonal billiards.

We remark that, when the limiting function $\overline{\mu}(S)$ is differentiable, the asymptotic level spacing distribution admits the following density:

$$P_{\overline{\mu}}(S) = \{[1-\overline{\mu}(S)]^2 + \overline{\mu}'(S)\}\exp\left[-\int_0^S [1-\overline{\mu}(\sigma)]d\sigma\right]. \quad (2.15)$$

III. EXAMPLE

As an example of the deviation from the Poisson distribution, we study the quantum systems whose energy levels are described by using positive integer numbers m and i as follows:

$$\epsilon_{m,i} = m^2 + \alpha i^2, \quad (3.1)$$

where α is the system parameter. Such energy levels are given, for instance, by the rectangular billiard system whose aspect ratio of two sides is characterized by α [18,24]. In this paper, the unfolding transformation of the energy levels $\{\epsilon_{m,i}\} \to \{\overline{\epsilon}_{m,i}\}$ (see Appendix) is done by using the leading Weyl term of the cumulative mean number $\mathcal{N}$ of energy levels,

$$\overline{\epsilon}_{m,i} \equiv \mathcal{N}(\epsilon_{m,i}) = \frac{\pi}{4\sqrt{\alpha}}\epsilon_{m,i}. \quad (3.2)$$

For a given energy interval $[\overline{\epsilon}, \overline{\epsilon}+\Delta\overline{\epsilon}]$, i or m can be regarded as an index that classifies energy levels into spectral components. In this paper, $i=1,2,3,\ldots,N$,

$$N = \left[\sqrt{\frac{4\sqrt{\alpha}(1+\gamma)\overline{\epsilon}-\pi}{\alpha\pi}}\right] \quad (3.3)$$

is adopted for classification where $\gamma \equiv \Delta\overline{\epsilon}/\overline{\epsilon}$, and $[x]$ stands for the maximum integer that does not exceed x. The relative weight of each component, ρ_i $(i=1,2,3,\ldots,N)$, is given by

$$\rho_i=\begin{cases}\dfrac{4(1+\gamma)}{N\pi\gamma\left(1+\dfrac{1}{\alpha N^2}\right)}\left[\sqrt{1+\dfrac{1-\alpha i^2}{\alpha N^2}}-\sqrt{\dfrac{1}{1+\gamma}+\dfrac{1}{\alpha N^2}\left(\dfrac{1}{1+\gamma}-\alpha i^2\right)}\right]+O\left(\dfrac{1}{N^2}\right) & \text{if } i<\sqrt{\dfrac{N^2+\dfrac{1}{\alpha}}{1+\gamma}} \\ \dfrac{4(1+\gamma)}{N\pi\gamma\left(1+\dfrac{1}{\alpha N^2}\right)}\sqrt{1+\dfrac{1-\alpha i^2}{\alpha N^2}}+O\left(\dfrac{1}{N^2}\right) & \text{if } \sqrt{\dfrac{N^2+\dfrac{1}{\alpha}}{1+\gamma}}\leqslant i\leqslant N.\end{cases} \tag{3.4}$$

As easily seen, ρ_i satisfies Assumption (i),

$$\max_i \rho_i \leqslant \frac{4}{\pi}\sqrt{1+\frac{1}{\gamma}}\frac{1}{N}\left(1+\frac{1}{\alpha N^2}\right)^{-1/2}+O\left(\frac{1}{N^2}\right) \to 0 \quad \text{as} \quad N\to+\infty. \tag{3.5}$$

Note that the limit of infinitely many components, $N\to+\infty$, corresponds to the high energy limit, $\bar{\epsilon}\to+\infty$ [see Eq. (3.3)], which is equivalent to the semiclassical limit. In this limit, the statistical weight of each subspectrum becomes sparse, since each element of $\mu(S;N)$ and $\rho_i\mu_i(S)$ tends to zero: $\rho_i\mu_i(S)\leqslant\max_j\rho_j\to 0$.

In the billiard system, each spectral component obeys a same scaling law (see Appendix), and thus is unfolded automatically by an overall unfolding procedure.

Figures 1(a)–1(c) show numerical results of the level spacing distribution $P(S)$ for three values of α. In case that α is far from rational, $P(S)$ is well approximated by the Poisson distribution [Fig. 1(a)]. While in case that α is close to a rational, $P(S)$ shows large deviation from the Poisson distribution [Figs. 1(b) and 1(c)]. When α is a rational expressed as $\alpha=p/q$, where p and q are coprime positive integers, $P(S)$ is not smooth and becomes a sum of δ functions in the limit of $\epsilon\to+\infty$ [2,9,18], which are separated by same step X,

$$X=\frac{\pi}{4\sqrt{pq}}.$$

Figures 2(a)–2(c) show the function $|\log_e[1-M(S;N)]|$ for three values of α corresponding to Figs. 1(a)–1(c), respectively. The dotted line in each figure corresponds to the Poisson distribution, $M(S)=1-\exp(-S)$. As shown in Figs. 2(b) and 2(c), the distribution function for small value of S clearly deviates from the Poisson distribution. However, for large value of S, it approaches a line whose slope is 1 (see the dashed line), and thus, the level spacings for large value of S obey the Poisson distribution.

In order to compare the non-Poissonian distributions and the classification given in the preceding section, we consider

$$D(S;N)=-\frac{1}{S}\log_e\left[1-\int_0^S[1-M(\sigma;N)]d\sigma\right]. \tag{3.6}$$

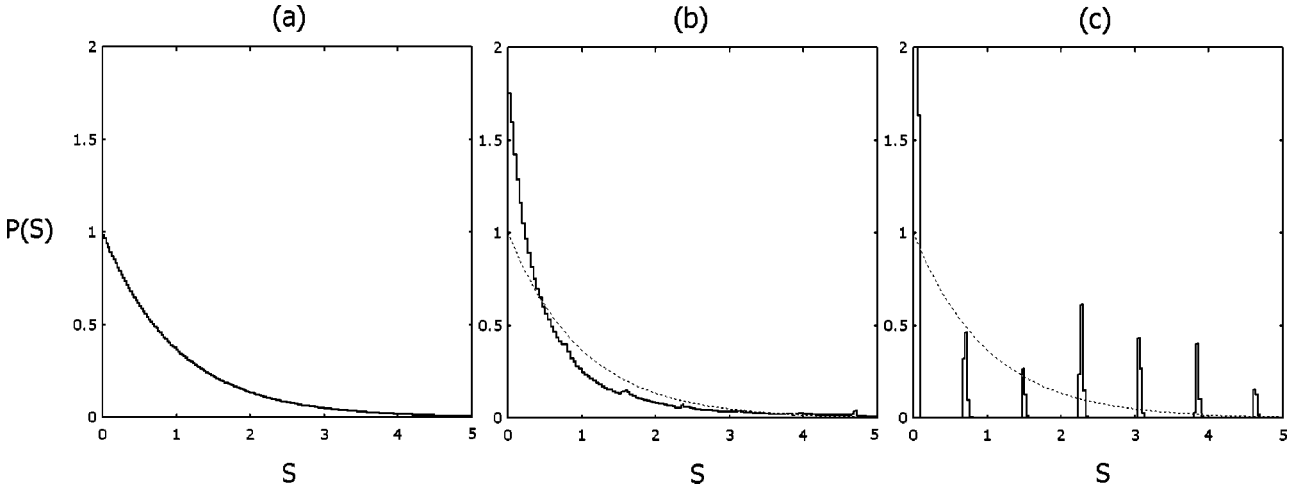

FIG. 1. Numerical results of the level spacing distribution $P(S)$ for various values of α: (a) $\alpha=1+(\pi/3)\times10^{-4}$, (b) $\alpha=1+(\pi/2)\times10^{-9}$, and (c) $\alpha=1+\pi\times10^{-11}$. We used energy levels $\bar{\epsilon}_{m,i}\in[300\times10^7,301\times10^7]$. Total numbers of levels are (a) 10 000 016, (b) 10 000 046, and (c) 10 000 043. The dotted curve in each figure shows the Poisson distribution, $P(S)=e^{-S}$.

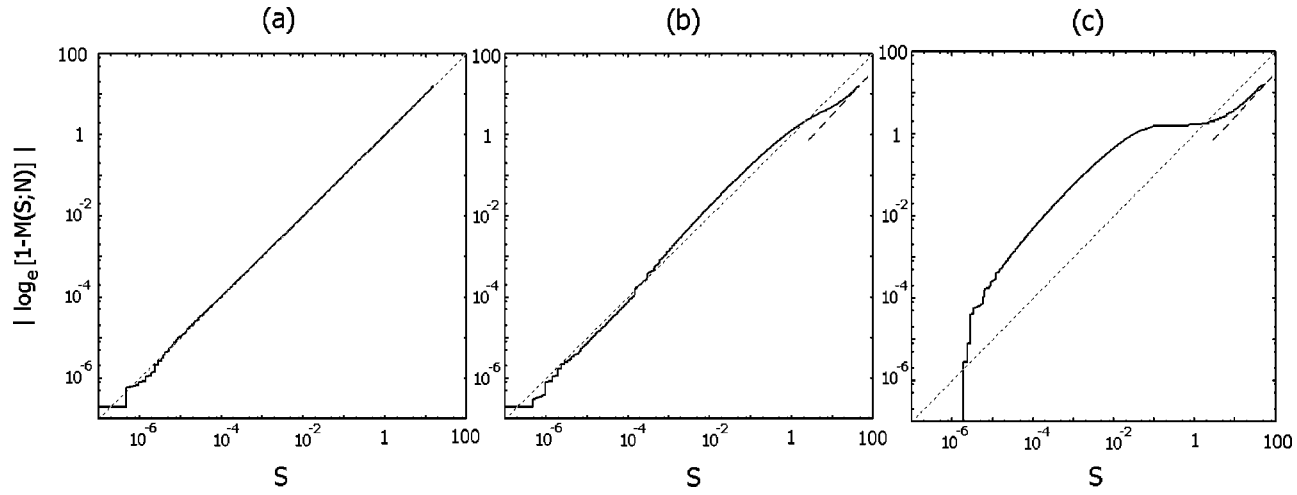

FIG. 2. Function $|\log_e[1-M(S;N)]|$ for (a) $\alpha=1+(\pi/3)\times10^{-4}$, (b) $\alpha=1+(\pi/2)\times10^{-9}$, and (c) $\alpha=1+\pi\times10^{-11}$. The dotted line in each figure corresponds to the Poisson distribution, $M(S)=1-\exp(-S)$.

When $N\to+\infty$, $D(S;N)$ approaches $(1/S)\int_0^S[1-\overline{\mu}(\sigma)]d\sigma$ and this function distinguishes the three cases as follows: In Case 1, i.e., where the level spacing obeys the Poisson distribution, $\lim_{N\to+\infty}D(S;N)=1$. In Case 2, $\lim_{N\to+\infty}D(S;N)\to c$ as $S\to+\infty$ $(c\neq1)$, and in Case 3, where the sub-Poisson distribution is expected, $\lim_{N\to+\infty}D(S;N)\to0$ as $S\to+\infty$.

Figure 3 shows $D(S;N)$ for different values of N. From this, one can think that $D(S;N)$ for $N=61905, S\leqslant10$ well approximates $\lim_{N\to+\infty}D(S;N)$.

Figure 4 shows $D(S;N)$ for the three values of α corresponding to Figs. 1(a)–1(c), respectively. In case that $P(S)$ is well characterized by the Poisson distribution [Fig. 1(a)], the corresponding function $D(S;N)$ agrees with 1. In case that $P(S)$ deviates from the Poisson distribution [Figs. 1(b) and 1(c)], $D(S;N)$ approaches a number less than 1 for $S\to+\infty$. Therefore, these results correspond to Case 2. In this model, we have not yet observed the clear evidence of Case 3. Such a case is expected when there is stronger accumulation of the energy levels of individual components.

IV. CONCLUSION AND DISCUSSION

In this paper, along the line of thoughts of Berry and Robnik [10], we investigated the level spacing distribution of systems with infinitely many independent components and discussed its deviations from the Poisson distribution. In the semiclassical limit, reflecting infinitely fine classical phase space structure, individual energy eigenfunctions are ex-

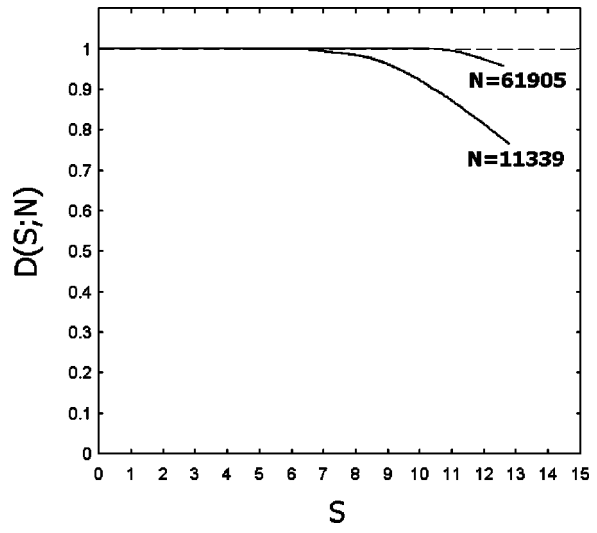

FIG. 3. $D(S;N)$ for $N=11\,339$ and for $N=61\,905$, which corresponds to energy levels $\overline{\epsilon}_{m,i}\in[100\times10^6,101\times10^6]$ and $\overline{\epsilon}_{m,i}\in[300\times10^7,301\times10^7]$, respectively. In each case, we fixed $\alpha=1+(\pi/3)\times10^{-4}$. The dashed line, $D(S)=1$, exhibits the Poisson distribution.

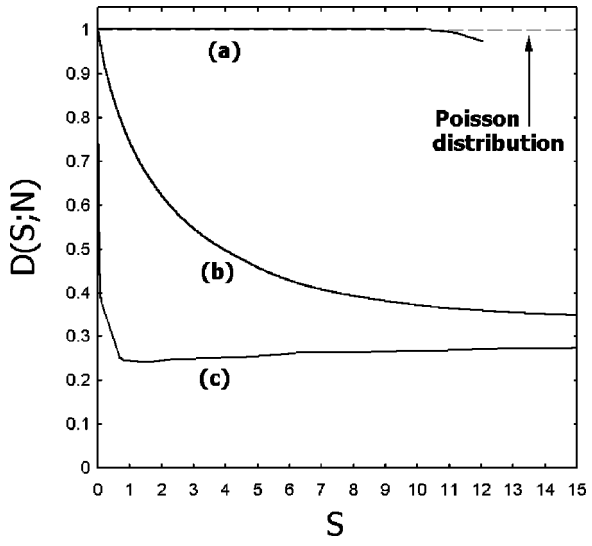

FIG. 4. $D(S;N)$ for (a) $\alpha=1+(\pi/3)\times10^{-4}$ and $N=61\,905$, (b) $\alpha=1+(\pi/2)\times10^{-9}$ and $N=61\,906$, and (c) $\alpha=1+\pi\times10^{-11}$ and $N=61\,906$, which are calculated from energy levels $\overline{\epsilon}_{m,i}\in[300\times10^7,301\times10^7]$. The dashed line, $D(S)=1$, exhibits the Poisson distribution.

pected to be well localized in the phase space and give independent contribution to the level statistics. Keeping this expectation in mind, we considered a situation where the system consists of infinitely many components and each of them gives an infinitesimal contribution. And by applying the arguments of Mehta, and Berry and Robnik, the limiting level spacing distribution was obtained, which is described by a single monotonically increasing function $\bar{\mu}(S)$ of the level spacing S. The limiting distribution is classified into three cases: Case 1, Poissonian if $\bar{\mu}(+\infty)=0$; Case 2, Poissonian for large S, but possibly not for small S if $0<\bar{\mu}(+\infty)<1$; and Case 3, sub-Poissonian if $\bar{\mu}(+\infty)=1$. Thus, even when energy levels of individual components are statistically independent, non-Poisson level spacing distribution is possible.

Note that the singular level spacing distribution can be taken into account in terms of nonsmooth cumulative distribution $M_{\bar{\mu}}$. Such a singularity is expected when there is strong accumulation of the energy levels of individual components. For certain class of systems, such accumulation is observable. One example is shown in Sec. III where the results show clear evidence of Case 2. Another example is studied by Shnirelman [19], Chirikov and Shepelyansky [25], and Frahm and Shepelyansky [26] for a certain types of systems which contain quasidegeneracy result from inherent symmetry (time reversibility). As is well known, the existence of quasidegeneracy leads to the sharp Shnirelman peak at small level spacings.

One of the interesting features of the level statistics is the level clustering, which is described by a nonvanishing value of the level spacing distribution function at $S=0$. Level clustering is expected for integrable systems or mixed systems, but not for strongly chaotic systems due to the level repulsion. For certain class of systems, rigorous results are available; Molchanov [7] analyzed the energy levels of a one-dimensional Schrödinger operator with random potential and showed that the level clustering arose from the localization of the eigenfunction in the semiclassical limit. Minami also analyzed a one-dimensional Schrödinger operator with δ potentials and reported the same result [27]. The limiting level spacing distribution obtained in this paper possesses level clustering property. Indeed, when $\bar{\mu}(S)$ is differentiable, the level spacing distribution function (2.15) has nonvanishing value at $S=0$: $P_{\bar{\mu}}(0)=1+\bar{\mu}'(0)>0$.

It is also interesting to extend the Berry-Robnik distribution (2.13) for the level statistics of the nearly integrable system with two degrees of freedom. The classical phase space of this system consists of regular and chaotic regions. Since the regular regions corresponding to the system consist of infinitely many independent regions, original proposal for the gap distribution by Berry and Robnik would be replaced by

$$E_{\bar{\mu}}(S;N)=\exp\left[-\rho_0\int_0^S[1-\bar{\mu}(\sigma)]d\sigma\right]\prod_{i=1}^{N}E_i^{\mathrm{RMT}}(S;\rho_i), \tag{4.1}$$

where $E_i^{\mathrm{RMT}}(S;\rho_i)$ is the gap distribution function of the ith chaotic component obtained from the random matrix theory. The above distribution formula is classified into the following cases: Case $1'$, $\bar{\mu}(+\infty)=0$, Berry-Robnik distribution; Case $2'$, $0<\bar{\mu}(+\infty)<1$, Berry-Robnik distribution for large S, but possibly not for small S; and Case $3'$, $\bar{\mu}(+\infty)=1$, a distribution function obtained by the superposition of spectral components obeying the sub-Poisson statistics and the random matrix theory. From this classification, one can see that formula (4.1) admits deviations from the Berry-Robnik distribution when $\bar{\mu}(+\infty)\neq 0$.

For nearly integrable systems with two degrees of freedom, Prosen and Robnik showed numerically that the Berry-Robnik formula (2.13) well approximates the level spacing distributions in the high energy region, which is called *the Berry-Robnik regime* [28], while it deviates in the low energy region and approximates the Brody distribution. They studied this behavior in terms of a fractional power dependence of the spacing distribution near the origin at $S=0$, which could be attributed to the localization properties of eigenstates on chaotic components [29,30]. From the above classification, one can see that condition $\bar{\mu}(+\infty)=0$ should be satisfied in the Berry-Robnik regime. While Case $2'$ and Case $3'$ in the above classification might propose another possibilities. When the spectral components corresponding to regular regions show strong accumulation, the level spacing statistics obeys the distribution formula (4.1) with $0<\bar{\mu}(+\infty)\leqslant 1$, and shows deviations from the Berry-Robnik distribution. Such possibilities will be studied elsewhere.

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APPENDIX: UNFOLDING OF SPECTRUM

The unfolding transformation of each energy level $\{\epsilon_n\}\rightarrow\{\bar{\epsilon}_n\}$ is done by using the cumulative mean number of levels up to energy ϵ [18],

$$\mathcal{N}([0,\epsilon])=\int_0^{\epsilon}d(x)dx. \tag{A1}$$

In the above equation, $d(\epsilon)$ exhibits the density of energy levels obtained by the Thomas-Fermi rule [5]:

$$d(\epsilon)=\frac{V(\epsilon)}{(2\pi\hbar)^f},\quad V(\epsilon)=\int\delta(\epsilon-H(\mathbf{q},\mathbf{p}))d^f\mathbf{q}d^f\mathbf{p}, \tag{A2}$$

where $(2\pi\hbar)^f$ is the Planck volume of the system with f degrees of freedom, $V(\epsilon)$ is the phase volume on the energy surface, $\delta(\epsilon-H)$ is the δ function, $H(\mathbf{q},\mathbf{p})$ is the classical Hamiltonian function, and $(\mathbf{q},\mathbf{p})$ are the coordinates and momenta in the phase space. The unfolding transformation of spectrum $\{\epsilon_n\}\rightarrow\{\overline{\epsilon_n}\}$ $(n=1,2,3,\ldots)$ is defined by

$$\overline{\epsilon_n}=\mathcal{N}([0,\epsilon_n]). \tag{A3}$$

Here, we consider the unfolding procedure of the ith subspectrum. Since the phase volume of the ith component is $\rho_i(\epsilon)V(\epsilon)$, the density of each subspectrum is then described by

$$d_i(\epsilon)=\frac{\rho_i(\epsilon)V(\epsilon)}{(2\pi\hbar)^f}=\rho_i(\epsilon)d(\epsilon). \tag{A4}$$

When the energy interval $[\epsilon,\epsilon+\Delta\epsilon]$ is sufficiently small, the phase-space geometry on the energy surface does not change, in general. In other words, $\rho_i(\epsilon)$ is approximated by a constant value in this ivterval, and the cumulative mean number of the ith subspectrum is thus described as

$$\mathcal{N}_i([\epsilon,\epsilon+\Delta\epsilon])=\frac{1}{(2\pi\hbar)^f}\int_{\epsilon}^{\epsilon+\Delta\epsilon}\rho_i(e)V(e)de \tag{A5}$$

$$\simeq\frac{\rho_i}{(2\pi\hbar)^f}\int_{\epsilon}^{\epsilon+\Delta\epsilon}V(e)de$$

$$=\rho_i\mathcal{N}([\epsilon,\epsilon+\Delta\epsilon]). \tag{A6}$$

Therefore, in the asymptotic limit $\Delta\epsilon\rightarrow 0$, one can see that each spectral component obeys a same scaling law and is unfolded automatically by an overall unfolding transformation:

$$\overline{\epsilon_n}=\mathcal{N}(\epsilon_n)=\frac{1}{\rho_i}\mathcal{N}_i(\epsilon_n). \tag{A7}$$

The billiard system is very convenient since the phase-space geometry does not change for variety of ϵ, and $\mathcal{N}_i(\epsilon)=\rho_i\mathcal{N}(\epsilon)$ is satisfactory even when the size of energy interval is not small.

[1] I.C. Percival, J. Phys. B **6**, L229 (1973).
[2] M.V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).
[3] O. Bohigas, M.J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[4] M.L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, San Diego, 1991).
[5] O. Bohigas, in *Random Matrices and Chaotic Dynamics*, 1989 Les Houches Session LII, Chaos and Quantum Physics, edited by M-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991).
[6] Ya.G. Sinai, Physica A **163**, 197 (1990).
[7] S.A. Molchanov, Commun. Math. Phys. **78**, 429 (1981); S.A. Molchanov, Math. USSR Izvestija **12**, 69 (1978).
[8] P.M. Bleher, J. Stat. Phys. **61**, 869 (1990); P.M. Bleher, *ibid.* **63**, 261 (1991).
[9] R.D. Connors and J.P. Keating, J. Phys. A **30**, 1817 (1997).
[10] M.V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
[11] V.I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1978); V.I. Arnold, Russ. Math. Surveys **18**, 685 (1963).
[12] W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd ed. (Wiley, New York, 1957).
[13] M.V. Berry, J. Phys. A **10**, 2083 (1977).
[14] M.V. Berry, Philos. Trans. R. Soc. London, Ser. A **287**, 237 (1977).
[15] B. Li and M. Robnik, J. Phys. A **28**, 4843 (1995).
[16] T. Prosen, Physica D **91**, 244 (1996).
[17] H. Makino, T. Harayama, and Y. Aizawa, Prog. Theor. Phys. Suppl. **139**, 477 (2000).
[18] M. Robnik and G. Veble, J. Phys. A **31**, 4669 (1998).
[19] A.I. Shnirelman, Usp. Mat. Nauk **30**, 265 (1975).
[20] B. Li and M. Robnik, J. Phys. A **27**, 5509 (1994).
[21] M. Robnik, Nonlinear Phenomena in Complex Systems **1**, 1 (1998).
[22] H. Makino, T. Harayama, and Y. Aizawa, Phys. Rev. E **63**, 056203 (2001).
[23] Ya.G. Sinai, *Kurs teorii veroyatnostej*, published by MGU, Theorem 13.1.
[24] G. Casati, B.V. Chirikov, and I. Guarneri, Phys. Rev. Lett. **54**, 1350 (1985).
[25] B.V. Chirikov and D.L. Shepelyansky, Phys. Rev. Lett. **74**, 518 (1995).
[26] K.M. Frahm and D.L. Shepelyansky, Phys. Rev. Lett. **78**, 1440 (1997).
[27] N. Minami, Prog. Theor. Phys. Suppl. **116**, (1994).
[28] M. Robnik and T. Prosen, J. Phys. A **30**, 8787 (1997).
[29] T. Prosen and M. Robnik, J. Phys. A **27**, L459 (1994).
[30] T. Prosen and M. Robnik, J. Phys. A **27**, 8059 (1994).